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PRIMENESS IN NEAR-RINGS WITH MULTIPLICATIVE SEMI-GROUP SATISFYING 'THE THREE IDENTITIES'

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Abstract

In this paper, the interconnections of completely prime, 3-prime and equiprime ideals are considered in right permutable, left permutable and medial near-rings. Some results for right self distributivity, left self distributivity and insertion factors in near-rings are given.

1. Introduction

Throughout this paper, all near-rings are right near-rings. This paper considers primeness in near-rings with the multiplicative semi-group satisfying one of the following identities:

a. $abc = acb$ (right permutable near-rings)

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b. $abc = bac$ (left permutable near-rings)

c. $abcd = acbd$ (medial near-rings).

Birkenmeier and Heatherly [6] called these "the three identities". They developed a theory of rings satisfying the three identities [5]. A perusal of the near-ring literature reveals many types of near-rings which satisfy one of the three identities. Pilz [12] used the phrase "weakly commutative" for "right permutability" in near-rings. Near-rings which are both right permutable and left permutable are called permutable. Also playing a role in this paper are the identities:

d. $abc = acbc$ (right self-distributive (RSD))

e. $abc = abac$ (left self-distributive (LSD)).

Kepka [11] investigated semi-groups which are LSD and Birkenmeier et al. [3] studied LSD rings.

An ideal I of a near-ring N is called a completely prime ideal of N if whenever $ab \in I$, then $a \in I$ or $b \in I$. The study of completely prime ideals in near-rings goes back at least to [13], where such an ideal is called a "prime ideal of type 2". The ideal I is said to be completely semi-prime if $a^2 \in I$ implies $a \in I$. In [13] Ramakotaiah and Rao defined the concept of a prime ideal of type 1. An ideal I of $N\,(I \lhd N)$ is prime of type 1 if for all $x, y \in N$ $xNy \subseteq I$ implies $x \in I$ or $y \in I$. Groenewald [8] used the phrase "3-prime ideal" for "prime ideal of type 1". An ideal I is a 3-semiprime ideal if whenever $xNx \subseteq I$, then $x \in I$. Booth et al. [7] gave another generalization of prime rings which they called equiprimeness. $P \lhd N$ is called equiprime if $a, x, y \in N$ $anx - any \in P$ for all $n \in N$ implies $a \in P$ or $x - y \in P$. If P is equiprime, then it is 3-prime. If the zero ideal of N is 3-prime (resp. completely prime, equiprime), then we say N is a 3-prime (resp., completely prime, equiprime) near-ring.

Birkenmeier and Heatherly [4] showed that 3-prime (3-semiprime) ideals in an LSD or RSD near-ring are also completely prime (completely semi-prime). In [6], these authors proved that 3-prime ideals in a medial near-ring are also completely prime.

The main aim of this paper is to find the conditions which satisfy 3-primeness implies equiprimeness in near-rings with the identities a, b, c, d and e.

For all undefined terms in near-rings, the reader may refer to Pilz [12].

2. Primeness in Near-Rings with Multiplicative Semi-Group Satisfying "The Three Identities"

In general, completely primeness doesn't imply equiprimeness. For example, if $(N, +)$ is any cyclic group of prime order $p(p > 2)$, define $ab = a$ if $b \neq 0$ and $ab = 0$ if $b = 0$, then N is a near-ring which is completely prime but not equiprime [7].

For a near-ring N, the distributive part of N is the set $\{d \in N : d \text{ is distributive}\}$ and denoted by N_d.

Lemma 2.1. *Let N be near-ring and* $P \lhd N$. *Then* $N_d P \subseteq P$.

Proof. Let $n_d \in N_d$. Then $n_d 0 = n_d(0 + 0) = n_d 0 + n_d 0$, i.e., $n_d 0 = 0$. Since $P \lhd N$, $n_d p = n_d(p + 0) - n_d 0 \in P$ for every $p \in P$.

Proposition 2.2. *Let N be a right permutable near-ring and* $P \lhd N$. *Then P is* 3*-prime if and only if P is completely prime.*

Proof. Note that for any near-ring a completely prime ideal is a 3-prime ideal. Assume N is right permutable, $xy \in P$ and P is a 3-prime ideal. Then $xyN^2 = xNyN \subseteq P$. Now either $x \in P$ or $yN \subseteq P$. If $yN \subseteq P$, then $yNy \subseteq P$. Hence $y \in P$. Thus P is completely prime.

Theorem 2.3. *Let N be a right permutable near-ring and let* $P \lhd N$ *be such that* $N_d \setminus P \neq \emptyset$. *If P is a completely prime ideal, then P is equiprime.*

Proof. Suppose that for $a, x, y \in N$, $anx - any \in P$ for every $n \in N$. Since N is right permutable, then $anx - any = axn - ayn = (ax - ay)n \in P$. Then $ax - ay \in P$ or $n \in P$, since P is completely prime. If $n \in P$, then $N = P$, which is a contradiction of choice of P because $N_d \setminus P \neq \emptyset$ and $N_d \setminus P \subseteq N$. Hence $ax - ay \in P$. By Lemma 2.1. for an $n_d \in N_d \setminus P$, $n_d(ax - ay) = n_d ax - n_d ay \in P$. Since N is right permutable, $n_d ax - n_d ay = n_d xa - n_d ya = n_d(x - y)a \in P$. Then $n_d(x - y) \in P$ or $a \in P$, since P is completely prime. Since $n_d \notin P$, $x - y \in P$ or $a \in P$. Therefore P is an equiprime ideal of N.

Corollary 2.4. *Let N be a right permutable near-ring and let $P \lhd N$ be such that $N_d \setminus P \neq \emptyset$. Then P is 3-prime if and only if P is equiprime.*

Proof. Since every equiprime ideal is also 3-prime, the proof is seen from Proposition 2.2 and Theorem 2.3.

Theorem 2.3. is illustrated by the following example:

Example (cf. [1]). Let the additive group $(\mathbb{Z}_6, +)$. Under a multiplication given in the following table, $(\mathbb{Z}_6, +, \cdot)$ is a right permutable near-ring. We say $N = (\mathbb{Z}_6, +, \cdot)$.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	1	5	3	1	5
2	0	2	4	0	2	4
3	3	3	3	3	3	3
4	0	4	2	0	4	2
5	3	5	1	3	5	1

Let $P = \{0, 3\}$. Then P is an ideal of N and $4 \in N_d \setminus P$, i.e., $N_d \setminus P \neq \emptyset$. Then N and P satisfy the conditions of Theorem 2.3. It is seen that P is completely prime and equiprime.

Proposition 2.5. *Let N be a left permutable near-ring and* $P \lhd N$ *be such that* $NP \subseteq P (or\, P^2 = P)$. *Then P is* 3*-prime if and only if P is completely prime.*

Proof. Assume N is left permutable, $xy \in P$ and P is a 3-prime ideal such that $NP \subseteq P$. Then $Nxy = xNy \subseteq P$. Thus P is completely prime (If $P^2 = P$, then for every $n \in N$ and for every $p \in P$, $np = np_1p_2 = p_1np_2 \in P$, i.e., $NP \subseteq P$).

Theorem 2.6. *Let N be a left permutable near-ring and let P be a proper ideal of N. If P is completely prime, then it is equiprime.*

Proof. For $a, x, y \in N$, assume $anx - any \in P$ for all $n \in N$. Since N is left permutable, then $nax - nay \in P$. Let $q \in N - P$. Then $(nax - nay)q = naxq - nayq = nxaq - nyaq = (nx - ny)aq \in P$. Since N is completely prime and $q \notin P$, then $(nx - ny)a \in P$. Now either $nx - ny \in P$ or $a \in P$. If $nx - ny \in P$, then $(nx - ny)q = nxq - nyq = xnq - ynq = (x - y)nq \in P$. Then $(x - y)Nq \subseteq P$. Since P is also 3-prime and $q \notin P$, then $x - y \in P$. Thus P is equiprime.

Corollary 2.7. *Let N be a left permutable near-ring and let P be a proper ideal of N such that* $NP \subseteq P (or\, P^2 = P)$. *Then P is* 3*-prime if and only if P is equiprime.*

Proof. It is seen from Proposition 2.5 and Theorem 2.6.

Proposition 2.8. *Let N be a permutable near-ring and* $P \lhd N$. *Then P is* 3*-prime if and only if P is equiprime.*

Proof. For $a, x, y \in N$, assume $anx - any \in P$ for all $n \in N$. Since N is permutable, then $anx - any = axn - ayn = xan - yan = xna - yna = (x - y)na \in P$. So $(x - y)Na \subseteq P$. Then either $x - y \in P$ or $a \in P$. Thus P is equiprime, the converse is straightforward.

Proposition 2.9 ([6, Proposition 2.7]). *Let N be a medial near-ring and $P \triangleleft N$. Then P is 3-prime if and only if P is completely prime.*

Theorem 2.10. *Let N be a medial near-ring and $P \triangleleft N$ be such that $N_d \setminus P \neq \emptyset$. If P is completely prime, then it is equiprime.*

Proof. For $a \in N - P$, $x, y \in N$, assume $anx - any \in P$ for all $n \in N$. Let $n_d \in N_d - P$. Since N is medial and $N_dP \subseteq P$ by Lemma 2.1, then $n_danx - n_dany = n_dnax - n_dnay \in P$. Then, $n_dnaxn_d - n_dnayn_d = n_dnxan_d - n_dnyan_d = (n_dnx - n_dny)an_d \in P$. Since P is completely prime and $an_d \notin P$, then $n_dnx - n_dny \in P$. Then $n_dnxn_d - n_dnyn_d = n_dxnn_d - n_dynn_d = n_d(x - y)nn_d \in P$. Then $n_d(x - y)\, Nn_d \subseteq P$. Since P is completely prime and $n_d \notin P$, then $x - y \in P$. Therefore P is equiprime.

Corollary 2.11. *Let N be a medial near-ring and $P \triangleleft N$ be such that $N_d \setminus P \neq \emptyset$. Then P is 3-prime if and only if P is equiprime.*

Proof. The proof is seen from Propositin 2.9 and Theorem 2.10.

$\beta_i(N)$ will denote the intersections of all i-prime ideals of N for $i = 3$, $e =$ equi, $c =$ completely. For more details on the prime radicals $\beta_i(N)$, the following papers: [2] and [9] are recommended. We have the following:

Corollary 2.12. *Let N be a zero-symmetric near-ring. If $\beta_c(N)$ is a 0-prime ideal of N, then*

(a) If N is a right permutable (or medial) near-ring and there exists a completely prime ideal P of N such that $N_d \setminus P \neq \emptyset$, then $\beta_e(N) \subseteq \beta_c(N) = \beta_3(N)$.

(b) If N is a left permutable near-ring and $N \neq P$ is a completely prime ideal of N, then $\beta_e(N) \subseteq \beta_c(N) = \beta_3(N)$.

(c) If N is a permutable near-ring, then $\beta_e(N) = \beta_c(N) = \beta_3(N)$.

Proof. (a) It is easily seen that $\beta_c(N)$ is a completely semi-prime ideal of *N*. Under assumptions $\beta_c(N)$ is a 0-prime ideal of *N*, then $\beta_c(N)$ is a completely prime ideal of *N* from [9]. Since *P* is a completely prime ideal of *N* such that $N_d \setminus P \neq \emptyset$ and $\beta_c(N) \subseteq P$, then $N_d \setminus \beta_c(N) \neq \emptyset$. Hence $\beta_c(N)$ is an equiprime ideal of *N* from Theorem 2.3 (for medial, from Theorem 2.10). Therefore $\beta_e(N) \subseteq \beta_c(N). \beta_c(N) = \beta_3(N)$ comes from Proposition 2.2 (for medial, from Proposition 2.9).

(b) $\beta_c(N)$ is a completely prime ideal of *N* by the proof of (a). Since $N \neq P$ is a completely prime ideal of *N* and $\beta_c(N) \subseteq P$, then $\beta_c(N) \neq N$. Hence $\beta_c(N)$ is an equiprime ideal of *N* from Theorem 2.6, i.e., $\beta_e(N) \subseteq \beta_c(N)$. Since *N* is a zero-symmetric near-ring, then $N\beta_c(N) \subseteq \beta_c(N)$. Therefore $\beta_c(N) = \beta_3(N)$ by Proposition 2.5.

(c) $\beta_c(N)$ is a completely prime ideal of *N* by the proof of (a). Hence the result follows from Proposition 2.8.

3. Insertion Factors, LSD and RSD Near-Rings

In a near-ring *N*, an element $x \in N$ is called an insertion factor in *N* if for every $a, b \in N$ $ab = axb$. Throughout this section, $\mathcal{I}$ denotes the set of all insertion factors in *N*. Every constant near-ring is an example of $\mathcal{I} = N$.

If $n \in N$, the annihilator of *n* is $(0 : n) = \{x \in N : xn = 0\}$.

Lemma 3.1. *Let N be a near-ring.*

(i) *If* $(0 : n) = 0$ *for all n in an* RSD *near-ring N, then* $\mathcal{I} = N$.

(ii) *If* $\mathcal{I} = N$, *then N is both* RSD *and* LSD.

(iii) *If N is* RSD, *then N has strong* IFP *property* [4, Lemma 2.8].

(iv) *If N is* RSD *and simple, then* $\mathcal{I} = N$.

(v) *If N is* RSD *and simple, then N is* LSD.

(vi) *Let* $\mathcal{I} = N$ *and* $P \triangleleft N$. *Then* P *is* 3-*prime if and only if* P *is completely prime.*

Proof. (i) Assume N is RSD and for every $n \in N(0 : n) = 0$. Then for every $a, b, x \in N$, $abx = axbx$. Then $ab - axb \in (0 : x) = 0$. Hence $\mathcal{I} = N$.

(ii) Suppose that $\mathcal{I} = N$. Then for every $a, b, c \in N$ $ab = acb$. Hence $abc = acbc$, i.e., N is RSD. Similarly N is LSD.

(iv) If N is RSD and simple, then either $(0 : n) = 0$ or $(0 : n) = N$ for all $n \in N$ as a result of (iii). If for all $n \in N(0 : n) = 0$, $\mathcal{I} = N$ by (i). If for all $n \in N(0 : n) = N$, then $ab = 0 = anb$ for all $a, b, n \in N$. Hence $\mathcal{I} = N$.

(v) From (iv) and (ii).

(vi) Assume $\mathcal{I} = N$, $xy \in P$ and P is a 3-prime ideal. Then $xy = xny \in P$ for every $n \in N$. So $xNy \subseteq P$. Thus P is completely prime. The converse is straightforward.

Theorem 3.2. *Let* $\mathcal{I} = N$ *and* P *an ideal of* N *such that* $N_d \setminus P \neq \emptyset$. *Then* P *is* 3-*prime if and only if* P *is equiprime.*

Proof. Assume P is a completely prime ideal of N and for $a \in N - P$, $x, y \in N$ $anx - any \in P$ for every $n \in N$. Since $N_d P \subseteq P$ by Lemma 2.1., then for $n_d \in N_d - P$ $n_d anx - n_d any \in P$. Since $P \triangleleft N$, $n_d anxn_d - n_d anyn_d \in P$. Since $\mathcal{I} = N$, then $n_d anxn_d - n_d anyn_d = n_d xanxn_d - n_d yanyn_d \in P$. By Lemma 3.1 (ii), N is LSD, then $x(an)xn_d = x(an)n_d$ and $y(an)yn_d = y(an)n_d$. Then $n_d anxn_d - n_d any n_d = n_d xanxn_d - n_d yanyn_d = n_d xann_d - n_d ann_d = n_d(x - y)ann_d \in P$. So $n_d(x - y)aNn_d \subseteq P$. Since P is completely prime and $a, n_d \notin P$, then $x - y \in P$. Thus P is equiprime. The remain of the proof is seen from Lemma 3.1 (vi).

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